Evaluation of Thresholds for Power Mean-based and Other Divisor Methods of Apportionment

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Abstract

For divisor methods of apportionment with concave up or concave down rounding functions, we prove explicit formulas for the threshold values—the lower and upper bounds for the percentage of population that are necessary and sufficient for a state to receive a particular number of seats. Among the rounding functions with fixed concavity are those based on power means, which include the methods of Adams, Dean, Hill-Huntington, Webster, and Jefferson. The thresholds for Dean’s and Hill-Huntington’s methods had not been evaluated previously. We use the formulas to analyze the behavior of the thresholds for divisor methods with fixed concavity, and compute and compare threshold values for Hill-Huntington’s method (used to apportion the U.S. House of Representatives).

Key words: thresholds, divisor methods, power means, Hill-Huntington’s method, concavity, U.S. House of Representatives

2000 MSC: 91B12, 91F10

1. Introduction

In the United States, apportionment most often arises in the context of the House of Representatives because Article I, Section 2 of the Constitution of the United States has been interpreted to mean that, after each decennial census, Members of Congress are allocated to each state based on the proportion of the state’s share of the population. Although Hill-Huntington’s method has been used for apportioning the U.S. House since 1941, Hamilton’s, Jefferson’s, and Webster’s methods have also been used to allocate representatives. Common to these and other apportionment methods is the property that the number of seats awarded to each state depends not only on its percentage of the population, but also on the distribution of the population among the remaining states. One way to express this variation is through thresholds of apportionment methods: the lower (resp. upper) threshold is the minimum (resp. maximum) percentage of the population a state may have to be awarded a specific number of representatives.

Thresholds have been studied primarily for their insight into parliamentary systems based on proportional representation in which apportionment is used to determine the number of seats awarded to each party. In this context, thresholds determine the lower and upper percentages of the votes each party must obtain to receive a fixed number of seats. Because specific apportionments take as inputs the house size (or total number of representatives to be apportioned), the number of states (or parties) to be represented and their populations and output the number of seats allocated to each state (or party), it follows that the lower and upper thresholds for each
number of seats allocated depend on the house size, the number of states, and the apportionment method used. Apportionment methods can be divided into two classes: largest remainder (e.g., Hamilton’s method) and divisor (or largest average) methods. The methods of Adams, Dean, Hill-Huntington, Webster, and Jefferson (the last two are referred to as Sainte-Laguë and d’Hondt in Europe, respectively) are all divisor methods, based on the generalized power mean for specific parameter values. Explicit formulas for thresholds have been determined for apportionment under a number of divisor methods, including Adams, Webster, and Jefferson, but not Dean and Hill-Huntington.

Rokkan (1968) first proposed the threshold of representation (later called the threshold of inclusion), the minimal percentage required to possibly gain one seat, and obtained expressions for this threshold for Hamilton’s, Jefferson’s and Webster’s methods. Rae et. al (1971) extended this analysis to thresholds of exclusion, the maximal percentage in which it is possible to be denied a seat. Lijphart and Gibberd (1977) generalized this work for thresholds for any number of seats, deriving values for Hamilton’s, Jefferson’s and Webster’s methods and a modified version of the Webster/Sainte-Laguë method. Oyama (1991) introduced a class of parametric apportionment methods with linear rounding function which generalized Adams’, Webster’s and Jefferson’s methods. Balinski and Ramirez (1999) determined bounds for the apportionment of each state under such parametric apportionment methods and demonstrated how these could be used to find the thresholds of inclusion and exclusion for this class of divisor methods—this was the first attempt at a more general approach to determine thresholds for a class of divisor methods.

For general divisor methods, Palomares and Ramirez (2003) re-posed the threshold question by demonstrating that lower and upper thresholds each solved an optimization problem. In general, the solutions to these optimization problems are not obvious and may require a combinatorially complex number of evaluations to determine the optimal values. Palomares and Ramirez (2003) also solved the optimization problems to determine the lower and upper thresholds for the linearly parameterized divisor methods.

In this paper, we solve the optimization problems of Palomares and Ramirez for divisor methods in which the associated rounding function has fixed concavity—either concave up or concave down. This result allows for explicit formulas for the thresholds for divisor methods based on power and geometric means, including Hill-Huntington’s and Dean’s methods. Because the linearly parameterized divisor methods considered by Palomares and Ramirez (2003) also have fixed concavity, our work may be viewed as a generalization. Other apportionment methods also have fixed concavity, including two apportionment methods proposed by Agnew (2008) that are based on the identric and logarithmic means.

Taking our cue from Palomares and Ramirez’ (2003) discussion of thresholds for parametric apportionment methods, we analyze some properties of thresholds for divisor methods with fixed concavity. We also compare thresholds among different methods in the case of the U.S. House of Representatives, and discuss their relationship to size bias. Additionally, we prove that Jefferson’s method is the only divisor method with fixed concavity that satisfies the majority property, in which a state (or party) representing at least half the population is guaranteed at least half the seats. Palomares and Ramirez (2003) had showed previously that Jefferson’s method was the only parameterized linear divisor method to satisfy this property.

Threshold values have been applied in the political science literature. For instance, Gallagher (1992) used his ranking of thresholds to characterize the propensity of apportionment methods to encourage parties to run together or separately. Taagepera (1998) related thresholds of inclusion and exclusion at the district level to thresholds at the national level; Taagepera (2002) looked
at the role of thresholds in the results of elections from 23 electoral systems. Also, Oyama and Ichimori (1995) examined several recent elections in Japan and used a number of different parametric apportionment results to compare size bias and to make recommendations about which method is optimal. Having explicit thresholds for a wider class of apportionment methods will allow for both broader theoretical and practical analyses of apportionment methods.

2. Preliminaries

Suppose that \( P \) people are distributed among \( n \) states, and that each state \( i \) has a population \( P_i \) where \( P_i \geq 0 \) and \( \sum_{i=1}^{n} P_i = P \). Further, suppose that \( H \) representatives are to be apportioned among the states based on the \( P_i \)'s. A divisor method corresponds to a rounding rule \( f \) which assigns to each positive integer \( h \), a value \( f(h) \in [h - 1, h] \) such that there do not exist positive integers \( a \) and \( b \) for which \( f(a) = a - 1 \) and \( f(b) = b \). The value \( f(h) \) acts as a bifurcation point in which numbers between \( h - 1 \) and \( f(h) \) are rounded down to \( h - 1 \) and numbers between \( f(h) \) and \( h \) are rounded up to \( h \). Under the apportionment method with rounding rule \( f \), state \( i \) is given \( h_i \) seats if there exists a divisor \( x > 0 \) such that \( P_i/x \) is in the interval \([f(h_i), f(h_i + 1)]\) for each \( i \) where \( \sum_{i=1}^{n} h_i = H \). For convenience, we define \( f(0) = 0 \). Hence state \( i \) receives no representatives if \( 0 \leq P_i/x \leq f(1) \). If \( P_i/x = f(h) \) for some integer \( h \), state \( i \) may be awarded either \( h - 1 \) or \( h \) representatives. If \( f(1) = 0 \), then each state with positive share of the population receives at least one representative as long as \( H > n \). If \( f(1) = 0 \) and \( H \leq n \), then by convention each of the \( H \) states with the largest \( P_i \) receive a representative.

In some cases, there may be more than one possible apportionment for a given set of populations. This happens, for instance, when a divisor \( x \) is such that \( P_i/x = f(h_i) \) and \( P_j/x = f(h_j) \). An \textit{a priori} rule may be used to break such a “tie,” in which state \( i \) may receive either \( h_i - 1 \) or \( h_i \) representatives and state \( j \) may receive either \( h_j - 1 \) or \( h_j \) representatives, as long as the house size requirement is met. From a formal perspective, apportionment under a divisor method (or any other method) is considered to be multi-valued. For a house size \( H \), let \( F_f(p_1, p_2, \ldots, p_n) \) denote the set of possible apportionments for the given distribution of population under the rounding rule \( f \) where \( p_i = P_i/P \).

Many well-known divisor methods have rounding functions that are derived from the family of power means defined by \( M_\alpha(x, y) = [(1/2) \cdot (x^\alpha + y^\alpha)]^{1/\alpha} \) for real numbers \( x \) and \( y \) and \( \alpha \in [-\infty, \infty] \) where the mean is defined in terms of limits for \( \alpha \in (-\infty, 0, \infty) \). Applied to \( x - 1 \) and \( 1 \), power means yield the set of rounding functions which includes the methods of Adams (\( \alpha = -\infty \)), Dean (harmonic mean; \( \alpha = -1 \)), Hill-Huntington (geometric mean; \( \alpha = 0 \)), Webster (arithmetic mean; \( \alpha = 1 \)), and Jefferson (\( \alpha = \infty \)); see, for example, Marshall, \textit{et al.} (2002), Dorfleitner and Klein (1999), and Nagaraja, \textit{et al.} (2008). Dorfleitner and Klein (1999) also examined the class of divisor methods that are based on generalized geometric means, \( G_f(x, y) = x^1 y^{1-\alpha} \) for \( \alpha \in [0, 1] \).

Table 1 lists the rounding functions for divisor methods based on power means, generalized geometric means, and the parameterized \( \gamma \) family analyzed by Palomares and Ramirez (2003). As the table indicates, Adams’, Webster’s and Jefferson’s method are also members of the \( \gamma \)-family while Adams’, Hill-Huntington’s, and Jefferson’s are also special cases of geometric means. Additionally, the table includes two divisor methods proposed by Agnew (2008) based on the identric and logarithmic means. Agnew shows that these apportionment methods can be viewed as solutions to natural optimization problems—the one for the identric mean, in particular, being equal to the likelihood ratio for the number of expected outcomes in a series of
multinomial trials. The rounding functions for these new methods are both defined at \( a = 1 \) by a limit: for the identric mean, \( f(1) = 1/e \) where \( e \) is Euler’s constant; for the logarithmic mean, \( f(1) = 0 \).

<table>
<thead>
<tr>
<th>Power mean</th>
<th>Geometric mean</th>
<th>( \gamma )-family</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = -\infty; \lambda, \gamma = 0 )</td>
<td>( (h - 1)h! )</td>
<td>( h - 1 + \gamma )</td>
</tr>
<tr>
<td>( h - 1 )</td>
<td>2( (h - 1)/(2h - 1) )</td>
<td>( \sqrt{(h - 1)}h )</td>
</tr>
</tbody>
</table>

Table 1: Rounding functions \( f(h) \) for some divisor methods.

Given \( n \) and \( H \), a natural question to ask is what range of \( p_i \) values may result in state \( i \) receiving \( h \) seats. The lower and upper thresholds, \( I_h \) and \( S_h \), respectively, are defined as follows. For a specific divisor method with rounding function \( f \), define

\[
I_h = \inf\{p_i | \text{there exist } p_2, \ldots, p_n \text{ such that } h_1 = h \text{ for some } (h_1, \ldots, h_n) \in F_f(p_1, \ldots, p_n) \} \\
S_h = \sup\{p_i | \text{there exist } p_2, \ldots, p_n \text{ such that } h_1 = h \text{ for some } (h_1, \ldots, h_n) \in F_f(p_1, \ldots, p_n) \}.
\]

The thresholds \( I_h \) and \( S_h \) are defined, for convenience, in terms of state 1, because divisor methods are symmetric with respect to permutations of the \( p_i \)'s. The interval \([I_h, S_h]\) denotes the range of percentages which any state may be awarded \( h \) seats. If \( p_i < I_h \), then state \( i \) is guaranteed fewer than \( h \) seats. If \( p_i > S_h \), then state \( i \) is guaranteed more than \( h \) seats. If \( I_h \leq p_i \leq S_h \), state \( i \) may or may not receive \( h \) seats, depending on the distribution of the other states. Of particular interest are: \( I_1 \), the threshold of inclusion, because if \( p_i < I_1 \), then state \( i \) receives no seats; \( S_0 \), the threshold of exclusion, because if \( p_i > S_0 \), state \( i \) must obtain at least 1 seat; and \( S_K \) for \( H = 2K + 1 \) because if \( p_i > S_K \), then state \( i \) is guaranteed a majority of the seats. The following proposition due to Palomares and Ramirez (2003) shows how \( I_h \) and \( S_h \) can be written as solutions to optimization problems in terms of the house size \( H \), the number of states \( n \), and the rounding function \( f \).

**Proposition 2.1.** (Palomares and Ramirez, 2003) Given a divisor method associated with a rounding function \( f(h) \),

\[
I_h = \min \frac{f(h)}{f(h) + \sum_{i=1}^n f(h_i + 1)} \text{ and } S_h = \max \frac{f(h + 1)}{f(h + 1) + \sum_{i=1}^n f(h_i)}
\]

where the optimas are taken over all nonnegative integers \( h_2, \ldots, h_n \) such that \( \sum_{i=1}^n h_i = H - h \).

For general rounding functions, we would expect that a state having roughly \( p_i = h/H \) of the population would receive \( h \) seats. Thus both \( I_h \) and \( S_h \) should be close to \( h/H \) with \( I_h \leq h/H \leq S_h \). To see this, note that by Eq. 1, \( I_h = \frac{1}{1 + \max \sum_{i=1}^n f(h_i + 1)f(h)} \). Since \( x - 1 \leq f(x) \leq x \) for all \( x \geq 1 \), we have

\[
\frac{H - h}{h} = \frac{\sum_{i=1}^n h_i}{h} \leq \frac{\sum_{i=1}^n f(h_i + 1)}{f(h)} \leq \frac{\sum_{i=1}^n (h_i + 1)}{h - 1} = \frac{H - h + n - 1}{h - 1}. 
\]
Thus
\[
\frac{h - 1}{H + n - 2} \leq I_h = \frac{1}{1 + \max \frac{\sum_{i=1}^{H} f(h+1)}{f(h)}} \leq \frac{1}{1 + \frac{H-h}{H}} = \frac{h}{H},
\]
so \(\frac{h-1}{H+n-2} \leq I_h \leq \frac{h}{H}\). Similar reasoning may be used to show that \(S_h \leq \frac{H+1}{H+n+1}\).

The proof of the proposition follows from induction on \(H\), which is essentially a restatement of the proposition for \(m = 2\).

3. Threshold expressions for rounding functions of fixed concavity

In general, determining explicit expressions for \(I_h\) and \(S_h\) depends on finding maxima and minima of sums of the form \(\sum_{i=1}^{H} f(h_i)\), which may require checking all the possible combinations of \(h_i\). However, if the rounding function is concave up or concave down, the extrema can be identified because it is possible to determine \(a\) \(\alpha\) priori which values of the \(h_i\)'s yield the extrema. Thus the values of \(I_h\) and \(S_h\) for apportionment methods whose rounding functions are of fixed concavity can be explicitly defined. This includes all the apportionment methods listed in Table 1. The rounding functions derived from the power means are concave down if \(\alpha > 1\) and concave up if \(\alpha < 1\) for all \(x \geq 1\). If \(\alpha = 1\), \(f(h) = h - 1/2\) is linear and hence both concave up and concave down; the rest are concave down, as are all the non-linear geometric means.

Note that rounding functions need only be defined on the nonnegative integers. However, for the convexity arguments that follow, we consider continuous extensions of rounding functions. A rounding function \(f\) can be extended continuously to a concave down function on \([1, \infty)\) if \(f(h+1) \geq (1/2) (f(h+2) + f(h))\) for positive integers \(h\). If the inequality is reversed, the rounding function can be extended continuously to a concave up function.

**Proposition 3.1.** Let \(f\) be a rounding function that for \(x \geq 1\) is increasing and satisfies \(f(x) \geq f(x-1) + f(1)\). Let \(h_1 + \cdots + h_m = M\) for some integers \(h_i \geq 0\). If \(f\) is concave down on \(x \geq 1\), then
\[
f(h_1) + \cdots + f(h_m) \geq \begin{cases} 
Mf(1) & \text{if } M < m \\
Mf(m-1) + (m-1)f(1) & \text{if } M \geq m, \text{ and}
\end{cases}
\]
where \(r \geq 0\) is defined by \(M + m = m \cdot \lfloor (M+m)/m \rfloor + r\). If \(f\) is concave up on \(x \geq 1\), then
\[
f(h_1) + \cdots + f(h_m) \geq r' f(\lfloor M/m \rfloor) + (m-r') f(\lfloor M/m \rfloor)\text{ and}
f(h_1) + \cdots + f(h_m) \leq f(M+1) + (m-1)f(1)
\]
where \(r' \geq 0\) is defined by \(M = m \cdot \lfloor M/m \rfloor + r'\).

The proof of the proposition follows from induction on \(m\) and relies on the following intuitive lemma, which is essentially a restatement of the proposition for \(m = 2\).
Lemma 3.2. Let \( f \) be an increasing and concave down function on \( x \geq 1 \). Let \( h_1, h_2 \geq 1 \) be integers such that \( h_1 + h_2 = M \). Then

\[
f(M - 1) + f(1) \leq f(h_1) + f(h_2) \leq f(\lceil M/2 \rceil) + f(\lfloor M/2 \rfloor)
\]

If \( f \) is concave up, the inequalities are reversed.

The proofs of Proposition 3.1 and Lemma 3.2 appear in the Appendix. Replacing the expressions from Proposition 3.1 into Eq. 1 yields the following result.

**Theorem 3.3.** Suppose \( f \) is a rounding function that is increasing, concave down and satisfies \( f(x) \geq f(x - 1) + f(1) \) on \( x \geq 1 \). Let \( r \) be defined by \( H - h + n - 1 = (n - 1) \cdot \lfloor x \rfloor + r \) where \( x = (H - h + n - 1)/(n - 1) \). Then

\[
I_h = \frac{f(h)}{f(h) + rf(\lfloor x \rfloor) + (n - 1 - r)f(\lfloor x \rfloor)} \quad \text{and}
\]

\[
S_h = \begin{cases} 
\frac{f(h) + rf(\lfloor x \rfloor) + (n - 1 - r)f(\lfloor x \rfloor)}{f(h) + f(H - h + 1) + (n - 2)f(1)} & \text{if } 0 \leq h \leq H - n + 1 \\
\frac{f(h + 1)}{f(h + 1) + rf(\lceil x \rceil) + (n - 1 - r')f(\lfloor x \rfloor)} & \text{if } H - n + 1 < h \leq H.
\end{cases}
\]

Suppose \( f \) satisfies the above properties but is concave up. Let \( r' \) be defined by \( H - h = (n - 1) \cdot \lfloor x' \rfloor + r' \) where \( x' = (H - h)/(n - 1) \). Then for all \( 0 \leq h \leq H \),

\[
I_h = \frac{f(h)}{f(h) + f(H - h + 1) + (n - 2)f(1)} \quad \text{and}
\]

\[
S_h = \frac{f(h + 1)}{f(h + 1) + rf(\lceil x' \rceil) + (n - 1 - r')f(\lfloor x' \rfloor)}.
\]

Since all the rounding functions listed in Table 1 satisfy the requirements of Theorem 3.3, it is possible to determine the associated thresholds by substituting their rounding functions into the equations above to obtain explicit expressions for \( I_h \) and \( S_h \) for each of the apportionment methods listed. As expected, if \( f(h) = h - 1 + \gamma \) is substituted into these formulas, the expressions obtained match those of Eq. 2.

**Example 3.4.** The U.S. House of Representatives uses Hill-Huntington’s method to apportion \( H = 435 \) Representatives among \( n = 50 \) states. The threshold functions for Hill-Huntington’s method are given by

\[
I_h(\text{HH}) = \frac{\sqrt{(h - 1)h}}{\sqrt{(h - 1)h + r \sqrt{\lceil x \rceil - 1} \lfloor x \rfloor + (n - 1 - r) \sqrt{\lfloor x \rfloor - 1} \lfloor x \rfloor}} \quad \text{and}
\]

\[
S_h(\text{HH}) = \frac{\sqrt{h(h + 1)}}{\sqrt{h(h + 1) + \sqrt{(H - h - n + 1)(H - h - n + 2)}}}
\]

where we use \( \text{HH} \) to denote Hill-Huntington’s method. Since \( f(1) = 0 \) for this method and each state has a positive share of the population, then each state will get at least 1 representative.

Following the 2000 census, New York saw its representation drop from 31 seats to 29 seats. Substituting \( h = 29 \) into these formulas, we obtain \( I_{29}(\text{HH}) = 0.0622 \) and \( S_{29}(\text{HH}) = 0.0761 \). New York’s proportion of the population in the 2000 Census was approximately 0.0675, which lies in the middle of the interval. However, a short series of calculations shows \( I_h(\text{HH}) \leq 0.0675 \) for \( h \leq 31 \) and that \( S_h(\text{HH}) \geq 0.0675 \) for \( h \geq 26 \). Hence, depending on the populations of the other states, New York could have received anywhere from 26 to 31 seats based on its 2000 population.
4. Properties and Comparisons of Thresholds

In general, the values of \( I_h \) and \( S_h \) as given by Theorem 3.3 behave as expected when considered as functions of \( H, n \), and \( h \). Both thresholds increase in \( h \) and decrease in \( H \). If \( n \) increases, then the value of \( I_h \) decreases while the value of \( S_h \) increases. Thus with more states, there is greater flexibility for the population of any one state while still retaining \( h \) seats—hence a larger interval for \([I_h, S_h]\).

Additionally, if the \( n \) states are roughly equal in size, the thresholds guarantee that each state receives roughly equal representation. In fact, for apportionment methods in the \( \gamma \)-family, \( I_h \leq 1/n \) (resp. \( S_h \leq 1/n \)) if and only if \( h < (H - 1)/n + 1 \) (resp. \( h < (H + 1)/n - 1 \)) and is equal to \( 1/n \) for \( h = (H - 1)/n + 1 \) (resp. \( h = (H + 1)/n - 1 \)). Thus if a state has \( 1/n \)th of the population, it will receive between \((H + 1)/n - 1\) and \((H - 1)/n + 1\) under a parametric apportionment method. General rounding functions that are either concave up or concave down have slightly looser bounds.

**Proposition 4.1.** Let \( H \geq n \) and \( f \) be an increasing function such that \( f(x) \geq f(x-1) + f(1) \) for all \( x \geq 1 \). If \( f \) is concave down, then

- \( I_h \leq 1/n \) if and only if \( h \leq (H - 1)/n + 1 \). In particular, if \( H \) and \( n \) are such that \( h = (H - 1)/n + 1 \) is an integer then \( I_h = 1/n \).
- \( S_h \leq 1/n \) if \( h \leq (H + 1)/n - 2 \). \( S_h \geq 1/n \) if \( h \geq (H + 1)/n \).

Similarly, if \( f \) is concave up, then

- \( S_h \leq 1/n \) if and only if \( h \leq (H + 1)/n - 1 \). In particular, if \( H \) and \( n \) are such that \( h = (H + 1)/n - 1 \) is an integer then \( S_h = 1/n \).
- \( I_h \leq 1/n \) if \( h \leq (H - 1)/n \). \( I_h \geq 1/n \) if \( h \geq H/n + 2 \).

The proof of the proposition follows from basic algebraic inequalities and the observation that, for example, if \( f \) concave down and \( h \) is an integer equal to \((H - 1)/n + 1\), then \( x = (H-h+n-1)/(n-1) = h\), so that \([x] = \lfloor x \rfloor = h \) and \( I_h = f(h)/f(h+1) = 1/n \). Details are in the Appendix. As a consequence of the proposition, if a state has \( 1/n \)th of the population and the rounding function is concave down, it will receive between \((H + 1)/n - 2\) seats and \((H - 1)/n + 1\) seats; if the rounding function is concave up, it will receive between \((H - 1)/n + 1\) and \((H + 1)/n\) seats.

A common characteristic of an appointment method is its degree of bias—the extent to which it favors large (or small) states. Among rounding functions of the form \( f(h) = h - 1 + \gamma \), Jefferson’s method most favors large states, Adams’ method most favors small states, and Webster’s method is relatively unbiased. (See Balinski and Young (2001) for a thorough analysis.) Gallagher (1992), less formally, suggested that \( I_1 \) and \( S_0 \) (the thresholds of inclusion and exclusion) could be used as an alternate way of comparing apportionment methods, since methods with low thresholds of inclusion and exclusion such as Adams’ method provide more representation for smaller states or parties.

Indeed for rounding functions in the \( \gamma \)-family, Palomares and Ramirez (2003) showed that for fixed \( H \) and \( n \), \( I_h \) and \( S_h \) are strictly increasing in \( \gamma \) if and only if \( h < (H - 1)/n + 1 \) and \( h < (H + 1)/n - 1 \), respectively. Thus, in particular, \( I_1(A) \leq I_1(W) \leq I_1(J) \) and \( S_0(A) \leq S_0(W) \leq S_0(J) \) where \( A, W, \) and \( J \) denote Adams’, Webster’s, and Jefferson’s methods, respectively. This agrees with Balinski and Young’s assessment of size bias.
Comparing thresholds of inclusion and exclusion among different apportionment methods is particularly important for analysis of proportional representation systems because they affect whether it is better for small parties to run separately or under joint platforms. In the context of the House of Representatives, however, it may be of more interest to compare thresholds for larger number of seats.

**Example 4.2.** Suppose \( H = 435 \) and \( n = 50 \). From Palomares and Ramirez (2003), the orderings of the thresholds \( I_h \) and \( S_h \) for the parametrized linear family reverse at \( h = \lfloor (H - 1)/n \rfloor = [9.68] = 9 \) and \( h = \lfloor (H + 1)/n - 1 \rfloor = [7.22] = 7 \), respectively. More generally, a short series of calculations shows that

\[
I_h(A) \leq I_h(D) \leq I_h(HH) \leq I_h(W) \leq I_h(J) \quad \text{if } 0 \leq h \leq 8
\]

\[
I_h(A) \leq I_h(HH) \leq I_h(D) \leq I_h(W) \leq I_h(J) \quad \text{if } h = 9
\]

\[
I_h(J) \leq I_h(W) \leq I_h(HH) \leq I_h(D) \leq I_h(A) \quad \text{if } 10 \leq h \leq 435.
\]

Similarly,

\[
S_h(A) \leq S_h(D) \leq S_h(HH) \leq S_h(W) \leq S_h(J) \quad \text{if } h = 0
\]

\[
S_h(A) \leq S_h(D) \leq S_h(W) \leq S_h(HH) \leq S_h(J) \quad \text{if } h = 1
\]

\[
S_h(A) \leq S_h(W) \leq S_h(D) \leq S_h(HH) \leq S_h(J) \quad \text{if } 2 \leq h \leq 3
\]

\[
S_h(A) \leq S_h(W) \leq S_h(J) \leq S_h(D) \leq S_h(HH) \quad \text{if } 4 \leq h \leq 7
\]

\[
S_h(J) \leq S_h(W) \leq S_h(A) \leq S_h(D) \leq S_h(HH) \quad \text{if } 8 \leq h \leq 9
\]

\[
S_h(J) \leq S_h(W) \leq S_h(A) \leq S_h(HH) \leq S_h(D) \quad \text{if } 10 \leq h \leq 128
\]

\[
S_h(J) \leq S_h(W) \leq S_h(HH) \leq S_h(A) \leq S_h(D) \quad \text{if } 129 \leq h \leq 192
\]

\[
S_h(J) \leq S_h(W) \leq S_h(HH) \leq S_h(D) \leq S_h(A) \quad \text{if } 193 \leq h \leq 435
\]

where the braced entries indicate what changes in the next inequality.

These orderings can be interpreted in several ways. For instance, if \( h = 29 \) as in Example 3.4, the largest value of \( I_h = 0.045 \) corresponds to Adams’ method; the smallest value of \( S_h = 0.0688 \) corresponds to Jefferson’s method. Since New York’s proportion of the population in 2000 was 0.0675, it lies between the lower and upper thresholds for \( h = 29 \) for all methods above. Hence New York could have received 29 seats under any of these methods. Notice also that for \( h \) near 29, the interval \([I_h(A), S_h(A)]\) lies inside \([I_h(HH), S_h(HH)]\), so there is a narrower window in which to receive an apportionment around 29 under Adams’ method than Hill-Huntington. In fact, under Adams’ method, NY could have received between 27 and 30 seats, a smaller range than the 26 to 31 seats under Hill-Huntington’ method determined in Example 3.4.

In the example above, the thresholds for Adams’ and Jefferson’s methods provide lower and upper bounds, respectively, for the thresholds of all other methods when \( h \) is small; when \( h \) is large, the bounds are reversed. Further, the orderings of the lower thresholds for \( h = 1 \) and \( h = H \) and of the upper thresholds for \( h = 0 \) and \( h = H - 1 \) appear to be monotonic in the power mean parameters. These observations are formalized below. The proof appears in the Appendix.
Proposition 4.3. For real numbers $\alpha_1 < \alpha_2$, let $I_h(\alpha_i)$ and $S_h(\alpha_i)$ be the thresholds associated with the divisor method based on the power mean with parameter $\alpha_i$. Further, let $I_h$ and $S_h$ be thresholds associated with any divisor method whose rounding function $f$ is increasing, has fixed concavity, and satisfies $f(x) \geq f(x-1) + f(1)$ on $x \geq 1$. It follows that

$$0 = I_1(A) < I_1(\alpha_1) < I_1(\alpha_2) < I_1(J) = \frac{1}{H + n - 1}$$

and $I_1(A) \leq I_1 \leq I_1(J)$.

$$0 = S_0(A) < S_0(\alpha_1) < S_0(\alpha_2) < S_0(J) = \frac{1}{H + 1}$$

and $S_0(A) \leq S_0 \leq S_0(J)$.

$$\frac{H}{H + n - 1} = I_H(J) < I_H(\alpha_2) < I_H(\alpha_1) < I_H(A) = 1$$

and $I_H(J) \leq I_H \leq I_H(A)$, and

$$\frac{H}{H + 1} = S_{H-1}(J) < S_{H-1}(\alpha_2) < S_{H-1}(\alpha_1) < S_{H-1}(A) = 1$$

and $S_{H-1}(J) \leq S_{H-1} \leq S_{H-1}(A)$.

Finally, we examine which apportionment methods satisfy the majority property—in which a state (or party) with at least half the population (or votes) is guaranteed at least half the seats. More precisely, if $H = 2K + 1$, an apportionment method satisfies the majority property if $S_K \leq 1/2$. Palomares and Ramirez (2003) showed that among rounding functions of the form $f(h) = h - 1 + \gamma$, this property is only held by Jefferson’s method where $f(h) = h$ for all $h$. Gallagher (1992) showed that this property is not held by either the modified Sainte-Lagüé method (Webster’s method except $f(1) = 0.7$) or Hamilton’s method. In fact, we show that among all either concave up or concave down rounding functions with $f(x) \geq f(x-1) + f(1)$ for $x \geq 1$ only Jefferson’s method satisfies the majority property. The proof of the following proposition appears in the appendix.

Proposition 4.4. Jefferson’s method is the only divisor method with either concave up or concave down rounding function that satisfies the majority property.

Appendix: Proofs of Results

We prove only the first part of Proposition 3.1 for concave down functions. The proof of the concave up case is similar. For concave down functions, the proof relies on Lemma 3.2.

Proof. (of Lemma 3.2) Since $f$ is concave down, $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$ for all $\lambda \in [0, 1]$ and $x, y \geq 0$. To show $f(h_1) + f(h_2) \geq f(M - 1) + f(1)$, let $x = M - 1$ and $y = 1$. Then

$$h_1 = \frac{h_1 - 1}{M - 2} (M - 1) + \frac{h_2 - 1}{M - 2} (1) \Rightarrow f(h_1) \geq \frac{h_1 - 1}{M - 2} f(M - 1) + \frac{h_2 - 1}{M - 2} f(1)$$

and

$$h_2 = \frac{h_2 - 1}{M - 2} (M - 1) + \frac{h_1 - 1}{M - 2} (1) \Rightarrow f(h_2) \geq \frac{h_2 - 1}{M - 2} f(M - 1) + \frac{h_1 - 1}{M - 2} f(1).$$

Summing the two inequalities gives $f(h_1) + f(h_2) \geq f(M - 1) + f(1)$.

To show $f(h_1) + f(h_2) \leq f([M/2]) + f([M/2])$, assume that the $h$ are ordered so that $h_2 \leq h_1$. Then $h_2 \leq [M/2] \leq [M/2] \leq h_1$, so there exists $\alpha$ and $\beta > 0$ such that

$$\alpha h_2 + (1-\alpha)h_1 = [M/2] \Rightarrow \alpha f(h_2) + (1-\alpha)f(h_1) \leq f([M/2])$$

and

$$\beta h_2 + (1-\beta)h_1 = [M/2] \Rightarrow \beta f(h_2) + (1-\beta)f(h_1) \leq f([M/2]).$$

Adding the two equations, we obtain $(\alpha + \beta)f(h_2) + (2 - \alpha - \beta)f(h_1) \leq f([M/2]) + f([M/2])$. Since $h_1 + h_2 = [M/2] + [M/2] = M$, there is always a solution in which $\alpha + \beta = 1$, where multiple solutions can occur only when $h_1 = h_2 = M/2$ for $M$ even. \qed
Proof. (of Proposition 3.1) Suppose first that \( h_i \geq 1 \) for all \( i \). We use induction on \( m \) to prove \( f(h_1) + \cdots + f(h_m) \geq f(M - m + 1) + (m - 1)f(1) \). Lemma 3.2 proves the \( m = 2 \) case. If the inequality is true for \( m - 1 \), then

\[
[f(h_1) + \cdots + f(h_{m-1})] + f(h_m) \geq f(M - h_m - m + 2) + (m - 2)f(1) + f(h_m)
\]

\[
= f(M - m + 1) + (m - 1)f(1)
\]

where the first line follows from the induction step, and the second line follows from the Lemma.

Let \( h_1 \geq h_2 \geq \cdots \geq h_m \) where \( h_i = 0 \) for \( m' < i \leq m \). Then since \( f(x) \geq f(x - 1) + f(1) \),

\[
f(h_1) + \cdots + f(h_m) = f(h_1) + \cdots + f(h_m) \geq f(M - m' + 1) + (m' - 1)f(1)
\]

\[
\geq f(M - m') + (m')f(1) \geq \cdots
\]

\[
\geq f(M - m' - k) + (m' + k)f(1) \geq \cdots
\]

If \( M < m \) then letting \( k = M - m' \), we have \( f(h_1) + \cdots + f(h_m) \geq f(1) + (M - 1)f(1) = Mf(1) \). If \( M \geq m \), letting \( k = m - m' - 1 \) gives \( f(h_1) + \cdots + f(h_m) \geq f(M - m + 1) - (m - 1)f(1) \).

To see that the second part of the proposition, note that it is equivalent to stating that \( f(h_1) + \cdots + f(h_m) \leq rf([M/m]) + (m - r)f([M/m]) \) for all \( h_i \geq 1 \) where \( r = (M + m - m' - 1)[M + m]/m \) such that the coefficients sum to \( m \) by \( M = r \cdot [M/m] + (m - r) \cdot [M/m] \). Now suppose that \( h_1 \geq h_2 \cdots \geq h_m \) are in decreasing order. By Lemma 3.2,

\[
f(h_1) + \cdots + f(h_m) \leq f(h_2) + \cdots + f(h_{m-1}) + [f([h_1 + h_m)/2]) + f([h_1 + h_m)/2])]
\]

\[
= f(h_1) + \cdots + f(h_{m-2}) + f([h_1 + h_m)/2]) + f([h_1 + h_m)/2])
\]

\[
+ [f([h_2 + h_{m-1})/2]) + f([h_2 + h_{m-1})/2])] \leq \cdots
\]

where each line is obtained by replacing the largest and smallest \( f(h_r) \) and \( f(h_r') \) values by the sum \( f([h_r + h_r')/2]) + f([h_r + h_r')/2]) \). Each time this is done, \( f(h_r) \) and \( f(h_r') \) get closer to each other until the right hand side consists of a sum of \( m \) terms of the form \( f([M/m]) \) and \( f([M/m]) \). Since \( M \) can be expressed uniquely as a linear combination of \( [M/m] \) and \( [M/m] \) such that the coefficients sum to \( m \) by \( M = r \cdot [M/m] + (m - r) \cdot [M/m] \), the result follows.

Proof. (of Proposition 4.1) To prove the first statement, suppose that \( f \) is concave down and that \( h \leq [(H - 1)/(n + 1)] \). Then

\[
x = (H - h + n - 1)/(n + 1) \geq (H - [(H - 1)/(n + 1)] + n - 1)/(n + 1) \geq (H - 1)/(n + 1).
\]

Thus \( |x| \geq [(H - 1)/(n + 1)] \) and \( |x| \geq |x| \geq h \). So \( I_h \leq \frac{f(h_0)}{[f([h + r]/(n + 1)])} \leq \frac{1}{h} \). Similar reasoning shows that \( I_h \geq 1/n \) if \( h \geq [(H - 1)/(n + 1)] \). The other statements are proved analogously.

Proof. (of Proposition 4.3) Substituting in \( f(1) \) and simplifying, the threshold of inclusion \( I_1(\alpha) \) for the divisor method with power mean rounding function \( f(h) = [(1/2) \cdot (h - 1)^\alpha + h^\alpha]^\beta \) is

\[
I_1(\alpha) = \frac{1}{1 + r ([|x|] - 1)^\alpha + [x]^\beta} + (n - 1 - r) [([|x|] - 1)^\alpha + [x]^\beta] \frac{1}{\beta},
\]

where \( r \) is defined by \( H + n - 2 = (n - 1) \cdot |x| + r \) and \( x = (H + n - 2)/(n - 1) \). \( I_1(\alpha) \) is increasing in \( \alpha \) because \( g(\alpha) = [(h - 1)^\alpha + h^\alpha]^\beta \) is decreasing in \( \alpha \) for \( h \geq 1 \). For \( \alpha_2 = \alpha_1 + \epsilon (\epsilon > 0) \), it follows that \( g(\alpha_1) - g(\alpha_2) > 0 \) because

\[
[(h - 1)^\alpha_1 + h^\alpha_1]^\beta - [(h - 1)^\alpha_1 + h^\alpha_1]^\beta[[(h - 1)^\alpha_1 + h^\alpha_1]^\beta - [(h - 1)^\alpha_1 + h^\alpha_1]^\beta] = [(h - 1)^\alpha_1 + h^\alpha_1]^\beta[(h - 1)^\alpha_1 + h^\alpha_1]^\beta - h^\alpha_1^\beta h^\alpha_1^\beta > 0.
\]
Proof. (of Proposition 4.4) Let $f$ be increasing and satisfy $f(x) \geq f(x - 1) + f(1)$ on $x \geq 1$. The claim of the proposition may be rewritten so that an apportionment method under a concave up or concave down $f$ satisfies the majority property if and only if $f(h) = h$.

Suppose that $f$ is concave down. Then $S_{K} \leq 1/2$ if and only if
\[ f(K + 1) \leq f(K - n + 3) + (n - 2)f(1) \quad \text{if} \quad K \geq n - 2 \quad \text{and} \quad f(K + 1) \leq (K + 1)f(1) \quad \text{if} \quad K \leq n - 2. \]

For the first condition, note that $f(K - n + 3) + (n - 2)f(1) \leq f(K - n + 4) + (n - 3)f(1) \leq \cdots \leq f(K + 1)$. Hence $f(K + 1) = f(K - n + 3) + (n - 2)f(1)$ for all $K \geq n - 2$. For the second condition, note that $f(K + 1) \geq f(K) + f(1) \geq \cdots \geq (K + 1)f(1)$. Hence $f(K + 1) = (K + 1)f(1)$ for all $K \leq n - 2$. Putting these together, we get $f(h) = h f(1)$ for all $h$. The only concave down, rounding function which satisfies this for all $h$ is $f(h) = h$.

If $f$ is concave up then $S_{K} \leq 1/2$ if and only if $f(K + 1) \geq rf([x]) + (n - 1 - r)f([x])$. But again $f(K + 1) \geq (K + 1)f(1)$, so $(K + 1)f(1) \leq f(K + 1) \leq rf([x]) + (n - 1 - r)f([x])$.

Now suppose $K = n - 2$. Then $x = (H - h)/(n - 1) = 1$ and the above inequality reduces to $(n - 1)f(1) \leq f(n - 1) \leq (n - 1)f(1)$. Hence, $f(n - 1) = (n - 1)f(1)$ and $f(h) = h$ for all $h$. \qed

References


